

# Multi-currency BGM Model

The Brace-Gatarek-Musiela (BGM) model is a multi-factor log-normal model. This model applies to both currencies. Its principle is to fix a tenor  $d$ , for instance 3 months, and to assume that each Libor rate at date  $t$ ,  $L(t, d)$  has a log-normal distribution in the “forward-neutral” probability of maturity  $t + d$ . The present model uses 4 factors, which we may assume independent.

Let  $F(t, T, d)$  be the FRA from  $T$  to  $T + d$  as observed at date  $t$ . The diffusion of  $F(t, T, d)$  with respect to  $t$  is given, as in the Heath-Jarrow-Morton (HJM) model, by a combination of factors:

$$dF(t, T, d) = F(t, T, d) \left( \mu(t, T) dt + \sum_{i=1}^4 \varphi_i(t, T) dW_i \right)$$

where the Brownian motions  $W_i$  are independent and the drift  $\mu$  is adapted to make the model arbitrage free. The factors  $\varphi_i(t, T)$  are defined by their “relative” size with respect to  $\varphi_1(t, T)$  and by a “volatility triangle”:

$$\varphi_i(t, T) = \psi_i (T - t) \sigma(t, T)$$

The “relative” factors  $\psi_i$  are parameterized in the following way:

$$\psi_1(t) = c \quad \psi_2(t) = \sqrt{\frac{1-c^2}{1+(t/a)^\alpha}}$$

$$\psi_3(t) = \sqrt{(1-c^2) \left(1 - \frac{1}{1+(t/b)^\beta}\right)}$$

$$\begin{aligned} \psi_4(t) &= \sqrt{1 - \psi_1(t)^2 - \psi_2(t)^2 - \psi_3(t)^2} \\ &= \sqrt{(1-c^2) \left( \frac{1}{1+(t/b)^\beta} - \frac{1}{1+(t/a)^\alpha} \right)} \end{aligned}$$

The sum of the squares of all the factors is imposed to be identically 1 so that the local volatility of each single Libor future rate is precisely  $\sigma(t,T)$ . The parameters  $a$ ,  $b$ ,  $c$ ,  $\alpha$  and  $\beta$  are input in the HJM spreadsheet under the following names:

$$\begin{array}{lll} c = \text{shift} & a = \text{sector 1} & b = \text{sector 2} \\ & \alpha = \text{slope 1} & \beta = \text{slope 2} \end{array}$$

The volatility triangle  $\sigma(t,T)$  bears this name because it is defined for  $t \leq T$ . It is discretized (see next sect.) and its values on the different cells are calibrated to match the market price of caplets and swaptions, except for diagonal values ( $t = T$ ) which are input by the user (the so-called “s-vols”).

A series of dates  $T_1 < \dots < T_n$ , spaced approximately by the tenor  $d$ , is fixed, for instance the IMM maturity dates. At each date  $t$ , the only available information is the short rate  $r(t)$ , the spot Libor rate  $L(t) = F(t, t, d)$ , the series of future rates  $F_i(t) = F(t, T_i, d)$ ,  $i = i_1(t), \dots, n$  where  $i_1(t)$  is the first index such that  $T_i > t$ , and the stub rate  $s(t)$ , which applies on the period  $[t, T_{i_1(t)}]$ . In practice, the time  $t$  is discretized and the short rate is that which applies over the time period of a diffusion step.

Discount factors are computed from the rates as follows:

$$DF(t, T_{i_1(t)}) = \frac{1}{1 + (T_{i_1(t)} - t) s(t)}$$

$$DF(t, T_{i+1}) = \frac{DF(T_i)}{1 + (T_{i+1} - T_i) F_i(t)} \quad i = i_1(t), \dots, n-1$$

Option values are discounted risk-neutral expectations of their pay-off. In a stochastic interest rate environment, the discounting should be taken as the accumulation of the spot rate  $r(t)$ . We define the *numeraire* at date  $t$  actualized in 0 as:

$$Num(t) = \exp\left(-\int_0^t r(u) du\right)$$

Again, a discretized version of this formula is used in practice (see sect. II).

The arbitrage theory states that:

$$\mathbf{E}_t \left( \frac{Num(T)}{Num(t)} \right) = DF(t, T)$$

The Bayesian rule implies:

$$\mathbf{E}_0 (Num(t) DF(t, T)) = DF(0, T)$$

The above structure applies to both currencies. In the sequel, superscripts  $d$  and  $f$  will specify whether we refer to domestic or foreign rates.

We denote by  $X(t)$  the exchange rate at date  $t$ . It follows a diffusion process defined by:

$$\frac{dX}{X} = (r^d - r^f) dt + \sigma_X(t) dW_X$$

Short rates  $r^d$  and  $r^f$  are processes described in the previous section.

The volatility  $\sigma_X$  can either be a time dependent parameter (deterministic volatility) or itself a process (stochastic volatility). In the latter case, its diffusion equation is:

$$d\sigma_X = \mu(t)(\bar{\sigma}(t) - \sigma_X) dt + \eta(t)\sigma_X dW_\sigma$$

This process is positive. As indicated, parameters  $\mu$ ,  $\bar{\sigma}$  and  $\eta$  can be time dependent. The first one is the *Mean Reversion*, the second one is the *Expected Spot Volatility* and the third one is the *Volatility of Volatility* or *Vvol*. The two Brownian motions  $W_x$  and  $W_\sigma$  can be correlated, with a time dependent correlation  $\rho^{x\sigma}(t)$ . Parameters  $\mu$ ,  $\bar{\sigma}$  and  $\eta$  must be positive, but  $\rho^{x\sigma}$  can have either sign.

In terms of volatility surface, the Vvol introduces a positive smile, the correlation induces a skew and the mean reversion makes the smile decrease with maturity. The expected spot volatility drives the term structure.

As mentioned previously, Brownian motions  $W_1^d, \dots, W_4^d$  are uncorrelated, as well as  $W_1^f, \dots, W_4^f$ . However, we allow  $W_i^d$  and  $W_i^f$  to be correlated with a possibly time dependent correlation  $\rho_i^{df}(t)$ . In order to avoid almost useless complexity, we assume that  $W_i^d$  and  $W_j^f$  are not correlated for non-equal indices  $i$  and  $j$ .

The Brownian motion  $W_x$  can be correlated to interest rate ones, with possibly time dependent correlations  $\rho_1^{Xd}(t), \dots, \rho_4^{Xd}(t), \rho_1^{Xf}(t), \dots, \rho_4^{Xf}(t)$ . In order to avoid non-positive definite correlation matrices, the correlation of  $W_\sigma$  with  $W_i^{d \text{ or } f}$  is set to the product  $\rho^{x\sigma}(t) \rho_i^{Xd \text{ or } Xf}(t)$ . If we decompose  $W_\sigma$  into a component totally correlated to  $W_x$  and another independent, then this is the same thing as saying that the independent component is also independent of the interest rate Brownian motions (ref <https://finpricing.com/lib/IrInflationCurve.html>).

A requirement for the global correlation matrix to be positive definite is that the four  $3 \times 3$  matrices made of correlations  $(\rho_i^{df}(t), \rho_i^{Xd}(t), \rho_i^{Xf}(t))$  be positive definite at all time. This is equivalent to saying that correlations lie in  $[-1,1]$  and that:

$$\rho_i^{df}(t)^2 + \rho_i^{Xd}(t)^2 + \rho_i^{Xf}(t)^2 \leq 1 + 2\rho_i^{df}(t)\rho_i^{Xd}(t)\rho_i^{Xf}(t)$$

This condition is necessary, but not sufficient. As the various interest rate factors in each currency are independent, one must have:

$$\sum_{i=1}^4 \rho_i^{Xd}(t)^2 \leq 1 \quad \text{and} \quad \sum_{i=1}^4 \rho_i^{Xf}(t)^2 \leq 1$$

These six inequalities should be sufficient for the correlation matrix to be positive definite.